

# Topologically Twisted $N = (2, 2)$ Supersymmetric Yang-Mills Theory on Arbitrary Discretized Riemann Surface

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## Abstract

We define supersymmetric Yang-Mills theory on an arbitrary two-dimensional lattice (polygon decomposition) while preserving one supercharge. When a smooth Riemann surface  $\Sigma_g$  with genus  $g$  emerges as an appropriate continuum limit of the generic lattice, the discretized theory becomes a topologically twisted  $\mathcal{N} = (2, 2)$  supersymmetric Yang-Mills theory on  $\Sigma_g$ . If we adopt the usual square lattice as a special case of the discretization, our formulation is identical with Sugino's lattice model. Although the tuning of parameters is generally required while taking the continuum limit, the number of necessary parameters is at most two because of the gauge symmetry and the supersymmetry. In particular, we do not need any fine-tuning if we arrange the theory so as to possess an extra global  $U(1)$  symmetry ( $U(1)_R$  symmetry) which rotates the scalar fields.

# 1 Introduction

Since the middle of the 1980s, after the first success of numerical QCD simulations based on lattice regularization, extension of the lattice technique to supersymmetric gauge theories has been pursued with great interest [1–4]. The hindrance encountered there was the fact that the regularization breaks the Poincaré invariance to its discrete subgroup and the supersymmetry cannot be straightforwardly realized on the lattice. To date, however, several lattice formulations of supersymmetric gauge theories have been developed by bypassing this difficulty. In particular, for one or two-dimensional theories with extended supersymmetries, there are such lattice formulations that are free from fine-tuning in taking the continuum limit thanks to partially preserved supercharges on the lattice.

In [5–18], some of the supercharges are exactly preserved on a hypercubic lattice by applying the so-called orbifolding procedure to supersymmetric matrix theory (mother theory)<sup>1</sup>. In these formulations, the bosonic link variables are not unitary but complex matrices, which restricts gauge groups to  $U(N)$  rather than  $SU(N)$ . In numerical simulations, therefore, we must introduce a large mass in the  $U(1)$  part of the complex link variables in order to fix the lattice spacing and take care of the fermionic zero modes in computing the Dirac matrix [20–22]. In [23–28], the authors discretized topologically twisted gauge theories while preserving one or two supercharges. In these formulations, lattice gauge fields are expressed by compact link variables on the hypercubic lattice, as in conventional lattice gauge theories and we can choose the gauge group  $SU(N)$ , which will be more convenient for numerical simulations [29–31]. In addition, the problem of the vacuum degeneracy of lattice gauge fields pointed out in these models [24] has recently been solved without using an admissibility condition [32].

As for three- and four-dimensional supersymmetric theories, apart from the formulations [33, 34] with exact chiral symmetry enabling the whole supersymmetry restoration in the continuum limit, lattice-regularized gauge theories require parameter tunings in taking the continuum limit even if part of supersymmetry is exactly preserved, since the symmetries on the lattice are generally insufficient to forbid relevant operators that break the rest symmetries<sup>2</sup>.

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<sup>1</sup> For a review, see [19].

<sup>2</sup> As another approach to circumvent this issue, four-dimensional  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory in the planar limit can be obtained by using a large- $N$  reduction technique which has been extensively studied from both the theoretical and numerical points of view [38–42]. As for theories with finite rank gauge group, a hybrid regularization has been proposed for four-dimensional  $\mathcal{N} = 2, 4$  supersymmetric Yang-Mills theories [43–45], where two different discretizations by lattice and matrix [35–37] are combined. For another numerical approach to  $\mathcal{N} = 4$  SYM, see [46–50].

As a common feature of lattice gauge theories so far, no attention has been paid to the topology of the spacetime. Indeed, all the previous lattice formulations of supersymmetric theories are discretized on a periodic hypercubic lattice; thus, the topology is always torus. Although this is natural because the main interest in conventional lattice gauge theories is in the UV nature, where the topology of the spacetime is usually irrelevant, it is also true that the topology is sometimes quite important for supersymmetric gauge theories especially in the context of topological field theory [51]. The importance of such theories has recently been increasing again, in relation to the height of the localization technique in supersymmetric gauge theories [52].

In this paper, we consider topologically twisted two-dimensional  $\mathcal{N} = (2, 2)$  supersymmetric Yang-Mills theory on a generic Riemann surface. We discretize the Riemann surface to an arbitrary lattice (polygons) and propose a way to define the supersymmetric gauge theory on it while preserving a supercharge. We show that we can define the theory on any decomposition of the two-dimensional surface and the tree-level continuum limit reproduces the continuum theory. We see that, if we consider the usual square lattice as a special case of discretization, our formulation coincides with Sugino's formulation [23–26]. We discuss that there are two types of theories depending on the hermiticity of the scalar fields: theories with and without an extra global  $U(1)$  symmetry ( $U(1)_R$  symmetry). If the theory has this symmetry, we can take the continuum limit without any fine-tuning, while we need one-parameter (two-parameter) tuning in taking the continuum limit if the theory does not have this symmetry and the gauge group is  $SU(N)$  ( $U(N)$ ).

This paper is organized as follows. In the next section, we briefly review the continuum topologically twisted two-dimensional  $\mathcal{N} = (2, 2)$  supersymmetric Yang-Mills theory on a curved background. In section 3, we define the theory on a general lattice and discuss the continuum limit and possible radiative corrections. The section 4 is devoted to the conclusion and discussion. In appendix A, we calculate the continuum limit of a face variable in detail.

## 2 Continuum two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory

We start with the two-dimensional  $\mathcal{N} = (2, 2)$  supersymmetric Yang-Mills theory on a flat Euclidean spacetime, which is obtained from a dimensional reduction of four-dimensional

$\mathcal{N} = 1$  supersymmetric Yang-Mills theory:

$$S = \frac{1}{2g_{2d}^2} \int d^2x \operatorname{Tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (\mathcal{D}_\mu \Phi) (\mathcal{D}_\mu \bar{\Phi}) + \frac{1}{4} [\Phi, \bar{\Phi}]^2 \right. \\ \left. + i\bar{\Psi} \Gamma_\mu \mathcal{D}_\mu \Psi - \frac{1}{2} \bar{\Psi} \Gamma_+ [\bar{\Phi}, \Psi] - \frac{1}{2} \bar{\Psi} \Gamma_- [\Phi, \Psi] \right\}, \quad (2.1)$$

where  $\mu, \nu = 1, 2$ ,  $\Gamma_\mu$  and  $\Gamma_\pm = \Gamma_3 \pm i\Gamma_4$  are four-dimensional Dirac matrices satisfying  $\{\Gamma_M, \Gamma_N\} = -2\delta_{MN}$  ( $M, N = 1, \dots, 4$ ),  $\Psi$  is a four-component spinor,  $\bar{\Psi} = -i\Psi^T \Gamma_4$ ,  $F_{\mu\nu}$  is the field strength of a gauge field  $A_\mu$ , and  $\Phi$  and  $\bar{\Phi}$  are scalar fields. We assume that the gauge group  $G$  is  $U(N)$  or  $SU(N)$  in the following.

We fix the notation of the gamma matrices by

$$\Gamma_1 = \begin{pmatrix} i\sigma_3 & \\ & i\sigma_3 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} i\sigma_1 & \\ & i\sigma_1 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} & -\sigma_2 \\ \sigma_2 & \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} & -i\sigma_2 \\ -i\sigma_2 & \end{pmatrix}, \quad (2.2)$$

and express the components of the spinor  $\Psi$  as

$$\Psi = (\lambda_1, \lambda_2, \chi, \eta/2)^T. \quad (2.3)$$

Then (2.1) reduces to

$$S = \frac{1}{2g_{2d}^2} \int d^2x \operatorname{Tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (\mathcal{D}_\mu \Phi) (\mathcal{D}_\mu \bar{\Phi}) + \frac{1}{4} [\Phi, \bar{\Phi}]^2 \right. \\ \left. + i\eta \mathcal{D}_\mu \lambda_\mu + 2i\chi (\mathcal{D}_1 \lambda_2 - \mathcal{D}_2 \lambda_1) + \lambda_\mu [\bar{\Phi}, \lambda_\mu] - \chi [\Phi, \chi] - \frac{1}{4} \eta [\Phi, \eta] \right\}. \quad (2.4)$$

We see that (2.1) (and of course (2.4)) is invariant under the supersymmetric transformation,

$$\delta\Phi = -i\bar{\xi} \Gamma_+ \Psi, \quad \delta\bar{\Phi} = -i\bar{\xi} \Gamma_- \Psi, \quad \delta A_\mu = -i\bar{\xi} \Gamma_\mu \Psi, \\ \delta\Psi = -F_{12} \Gamma_{12} \xi - \frac{1}{2} (\mathcal{D}_\mu \bar{\Phi}) \gamma_{\mu+} \xi - \frac{1}{2} (\mathcal{D}_\mu \Phi) \Gamma_{\mu-} \xi - \frac{i}{4} [\Phi, \bar{\Phi}] \Gamma_{+-} \xi, \quad (2.5)$$

where  $\xi$  is a four-component spinor parameter and  $\Gamma_{MN} \equiv \frac{1}{2} [\Gamma_M, \Gamma_N]$ .

Now let us consider a specific SUSY transformation associated with the parameter  $\xi = (0, 0, 0, \epsilon)^T$  and define the corresponding supercharge  $\hat{Q}$  as<sup>3</sup>

$$\delta\phi \equiv -i\epsilon \left( \hat{Q}\phi \right), \quad (2.6)$$

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<sup>3</sup> Here we have put a hat on the supercharge  $Q$  in order to distinguish it from the one appeared in the discretized theory in the next section.

for an arbitrary field  $\phi$ . We can read off the  $\hat{Q}$ -transformation of the fields as

$$\begin{aligned}\hat{Q}\Phi &= 0, \\ \hat{Q}\bar{\Phi} &= \eta, & \hat{Q}\eta &= [\Phi, \bar{\Phi}], \\ \hat{Q}A_\mu &= \lambda_\mu, & \hat{Q}\lambda_\mu &= iD_\mu\Phi, \\ \hat{Q}Y &= [\Phi, \chi], & \hat{Q}\chi &= Y,\end{aligned}\tag{2.7}$$

where  $Y$  is an auxiliary field. Then the action (2.4) can be expressed in the  $\hat{Q}$ -exact or topologically twisted form [51, 53] by

$$S = \hat{Q} \frac{1}{2g_{2d}^2} \int d^2x \operatorname{Tr} \left[ \frac{1}{4} \eta [\Phi, \bar{\Phi}] - i\lambda^\mu D_\mu \bar{\Phi} + \chi (Y - 2iF_{12}) \right]. \tag{2.8}$$

It is important that the  $\hat{Q}^2$  is equal to the infinitesimal gauge transformation with a parameter  $\Phi$ . Since  $\hat{Q}$  is acting on a gauge-invariant expression in the action (2.8), the  $\hat{Q}$ -invariance of the action is manifest.

We next extend the above theory to that on a curved background. One of the motivations for considering topological twist is to preserve a partial supersymmetry in a curved background [54]. The supersymmetry we usually use is completely broken on a curved background because there is in general no covariantly constant spinor. However, by twisting the local Lorentz symmetry with R symmetry, there can appear “scalar supercharges” which are preserved in any curved background. The supercharge  $\hat{Q}$  in (2.8) becomes the scalar supercharge as it is, and thus we can define topological Yang-Mills theory on the curved spacetime while keeping  $\hat{Q}$  as

$$S = \hat{Q} \frac{1}{2g_{2d}^2} \int_{\Sigma_g} d^2x \sqrt{g} \operatorname{Tr} \left[ \frac{1}{4} \eta [\Phi, \bar{\Phi}] - ig^{\mu\nu} \lambda_\mu D_\nu \bar{\Phi} + \chi (H - 2if) \right], \tag{2.9}$$

where the covariant derivative  $D_\mu$  now includes not only the gauge field but also the spacetime connection,  $\hat{Q}$  is the same as in (2.7),  $\Sigma_g$  is an oriented or unoriented two-dimensional manifold with the metric  $g_{\mu\nu}$ <sup>4</sup> and  $f(x) = \frac{1}{2} \frac{\epsilon^{\mu\nu}}{\sqrt{g(x)}} F_{\mu\nu}(x)$  is the Poincaré dual of the field strength. Because of the deformation of the background, the other three supersymmetries are broken in general.

Here we make some comments. First, the operations of twisting and curving do not commute. The action (2.9) is obtained by twisting the theory on the flat spacetime followed by curving the background. This theory differs from the one obtained by first curving the background followed by twisting (or renaming the fermionic fields). In the following section, we discretize the former (topological) theory. Therefore, even if we take

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<sup>4</sup>  $\Sigma_g$  can have even boundaries. In that case, we take the free boundary condition for simplicity.

the continuum limit, we do not obtain the latter (physical) theory. We note that it does not conflict with the fact that the continuum limit of Sugino's lattice formulation is the physical supersymmetric gauge theory [23–26]. This is because Sugino's formulation is defined on a flat spacetime where the physical theory and the topological theory coincide and twisting is merely a renaming of the fields.

Second, we can choose the hermiticity of the scalar fields  $\Phi(x)$  and  $\bar{\Phi}(x)$ . They are usually regarded as hermitian conjugate with each other from the construction; they are originally related to the components of the gauge fields of the four-dimensional theory as  $\Phi = A_3 + iA_4$  and  $\bar{\Phi} = A_3 - iA_4$ . In this case, the theory possesses  $U(1)_R$  symmetry,

$$\begin{aligned}\Phi &\rightarrow e^{2i\alpha}\Phi, & \bar{\Phi} &\rightarrow e^{-2i\alpha}\bar{\Phi}, & A_\mu &\rightarrow A_\mu, \\ \eta &\rightarrow e^{-i\alpha}\eta, & \lambda_\mu &\rightarrow e^{i\alpha}\lambda_\mu, & \chi &\rightarrow e^{-i\alpha}\chi.\end{aligned}\tag{2.10}$$

On the other hand, as often adopted in the context of the topological field theory, we can instead regard  $\Phi(x)$  and  $\bar{\Phi}(x)$  as independent hermitian variables. As a result, it is impossible to impose the  $U(1)$  rotation (2.10). This choice completely changes the theory. For example, the expectation value  $\langle \int d^2x \sqrt{g(x)} \text{Tr}(\Phi(x)^n) \rangle$  is zero in the former theory because of the  $U(1)_R$  symmetry (2.10) but it takes some non-trivial value in the latter theory. We can consider both theories depending on the purpose and can use the same discretization, explained in the next section.

### **3 $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory on an arbitrary discretized Riemann surface**

In this section, we discretize the continuum theory described in the previous section on a given decomposition of the two-dimensional surface, i.e., a set of sites, links and faces. As mentioned in the previous section, we can use the same discretization if we regard the scalar field  $\Phi$  as either complex or hermitian so we do not specify it in constructing the discretized formulation. We will see, however, that this choice is crucial in considering radiative corrections.

### 3.1 Definition of the model

A polygon decomposition of the two-dimensional surface consists of a set of sites  $S$ , links  $L$  and faces  $F$ , respectively:

$$\begin{aligned} S &\equiv \{s | s = 1, \dots, N_S\}, \\ L &\equiv \{\langle st \rangle | s, t \in S\}, \\ F &\equiv \{(s_1, \dots, s_n) | s_1, \dots, s_n \in S, (s_i, s_{i+1}) \in L \text{ or } (s_{i+1}, s_i) \in L\}, \quad (s_{n+1} \equiv s_1), \end{aligned} \quad (3.1)$$

where  $N_S$  is the number of sites, a link  $\langle st \rangle$  possesses a direction from  $s$  to  $t$ , and a face  $(s_1, \dots, s_n)$  is a surface surrounded by the links  $\langle s_i s_{i+1} \rangle$  ( $i = 1, \dots, n$ )<sup>5</sup>. We sometimes call the first site  $s_1$  of the face  $f \equiv (s_1, \dots, s_n)$  as the representative point (site) of the face  $f$ . This is apparently a generalization of the usual square lattice which is given by the data,

$$\begin{aligned} S &= \{\vec{X} = (x, y) | 1 \leq x \leq L_x, 1 \leq y \leq L_y\}, \\ L &= \left\{ \left\langle \vec{X} \vec{X} + \hat{x} \right\rangle, \left\langle \vec{X} \vec{X} + \hat{y} \right\rangle | \vec{X} \in S \right\}, \\ F &= \left\{ \left( \vec{X}, \vec{X} + \hat{x}, \vec{X} + \hat{x} + \hat{y}, \vec{X} + \hat{y} \right) | \vec{X} \in S \right\}. \end{aligned} \quad (3.2)$$

We next consider the following “fields” associated with the sites, links and faces of a given decomposition, respectively:

$$\begin{aligned} \Phi_s, \bar{\Phi}_s, \eta_s &: \text{site variables } (s \in S), \\ U_{st}, \Lambda_{st} &: \text{link variables } (\langle st \rangle \in L), \\ Y_f, \chi_f &: \text{face variables } (f \in F), \end{aligned} \quad (3.3)$$

where  $\Phi_s, \bar{\Phi}_s, U_{st}$  and  $Y_f$  are bosonic variables and  $\eta_s, \Lambda_{st}$  and  $\chi_f$  are fermionic variables. We assume that the site variables  $\Phi_s, \bar{\Phi}_s$  and  $\eta_s$  live on the site  $s$ , the link variables  $U_{st}$  and  $\Lambda_{st}$  live on the link  $\langle st \rangle$ , and the face variables  $Y_f$  and  $\chi_f$  live on the representative point of the face  $f$ . We often express the link fermion  $\Lambda_{st}$  as

$$\Lambda_{st} \equiv \lambda_{st} U_{st}, \quad (3.4)$$

where  $\lambda_{st}$  lives on the site  $s$ . We assume that  $U_{st} \in G$  and the other fields including  $\lambda_{st}$  are in the adjoint representation of  $G$ . For a given link  $\langle st \rangle$ , we sometimes use the

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<sup>5</sup> Only the sites  $s_i$  and  $s_{i+1}$  ( $i = 1, \dots, n$ ) must be connected by a link.

notation  $U_{ts} \equiv U_{st}^{-1}$ . Then the gauge transformations of the fields are

$$\begin{aligned}\Phi_s &\rightarrow g_s \Phi_s g_s^{-1}, & \bar{\Phi}_s &\rightarrow g_s \bar{\Phi}_s g_s^{-1}, & \eta_s &\rightarrow g_s \eta_s g_s^{-1}, \\ U_{st} &\rightarrow g_s U_{st} g_t^{-1}, & \Lambda_{st} &\rightarrow g_s \Lambda_{st} g_t^{-1}, \\ Y_f &\rightarrow g_f Y_f g_f^{-1}, & \chi_f &\rightarrow g_f \chi_f g_f^{-1},\end{aligned}\tag{3.5}$$

where  $g_s \in G$  ( $s \in S$ ) and we have used the same symbol  $f$  to describe a face and the representative point in the last line. It is easy to see that  $\lambda_{st}$  transforms as  $\lambda_{st} \rightarrow g_s \lambda_{st} g_s^{-1}$  under the gauge transformation.

Corresponding to the SUSY transformation (2.7), we consider the following transformation of the fields on the general lattice:

$$\begin{aligned}Q\Phi_s &= 0, \\ Q\bar{\Phi}_s &= \eta_s, & Q\eta_s &= [\Phi_s, \bar{\Phi}_s], \\ QU_{st} &= i\lambda_{st}U_{st}, & Q\lambda_{st} &= i(U_{st}\Phi_t U_{st}^{-1} - \Phi_s + \lambda_{st}\lambda_{st}), \\ QY_f &= [\Phi_f, \chi_f], & Q\chi_f &= Y_f.\end{aligned}\tag{3.6}$$

Note that the third line can be rewritten as

$$QU_{st} = i\Lambda_{st}, \quad Q\Lambda_{st} = i(U_{st}\Phi_t - \Phi_s U_{st}),\tag{3.7}$$

in terms of  $\Lambda_{st}$  instead of  $\lambda_{st}$ . It is easy to see that  $Q^2$  is equal to the infinitesimal gauge transformation with the parameter  $\Phi_s$ ; thus,  $Q$  is nilpotent if it acts on a gauge-invariant expression. Using this supercharge, we define the action,

$$\begin{aligned}S &= S_S + S_L + S_F \\ &\equiv Q \sum_{s \in S} \alpha_s \Xi_s + Q \sum_{\langle st \rangle \in L} \alpha_{\langle st \rangle} \Xi_{\langle st \rangle} + Q \sum_{f \in F} \alpha_f \Xi_f,\end{aligned}\tag{3.8}$$

with

$$\Xi_s \equiv \frac{1}{2g_0^2} \text{Tr} \left\{ \frac{1}{4} \eta_s [\Phi_s, \bar{\Phi}_s] \right\},\tag{3.9}$$

$$\Xi_{\langle st \rangle} \equiv \frac{1}{2g_0^2} \text{Tr} \left\{ -i\lambda_{st} (U_{st}\bar{\Phi}_t U_{st}^{-1} - \bar{\Phi}_s) \right\},\tag{3.10}$$

$$\Xi_f \equiv \frac{1}{2g_0^2} \text{Tr} \left\{ \chi_f (Y_f - i\beta_f \mu(U_f)) \right\},\tag{3.11}$$

where  $\alpha_s$ ,  $\alpha_{\langle st \rangle}$ ,  $\alpha_f$  and  $\beta_f$  are constants that will be fixed later so that the theory has an



appropriate continuum limit,  $\mu(U_f)$  is given by [32]

$$\mu(U_f) = \begin{cases} 2i \left[ (U_f - U_f^{-1})^{-1} (2 - U_f - U_f^{-1}) \right. \\ \quad \left. + (2 - U_f - U_f^{-1}) (U_f - U_f^{-1})^{-1} \right] & \text{for } G = U(N), \\ \frac{2i}{M} \left[ (U_f^M - U_f^{-M}) (2 - U_f^M - U_f^{-M}) \right. \\ \quad \left. + (2 - U_f^M - U_f^{-M}) (U_f^M - U_f^{-M}) \right] & \text{for } G = SU(N), \end{cases} \quad (3.12)$$

with  $2M > N$ , and  $U_f$  is the “plaquette variable” defined by

$$U_f \equiv \prod_{i=1}^n U_{s_i s_{i+1}}, \quad (3.13)$$

for  $f = (s_1, \dots, s_n)$ . Note that the form of  $\mu(U_f)$  is determined in order that the theory possesses unique vacuum at  $U_f = 1$  (see [32] for details). The explicit expression of the action is

$$S = S_b + S_f, \quad (3.14)$$

with

$$\begin{aligned} S_b = & \frac{1}{2g_0^2} \sum_{s \in S} \alpha_s \text{Tr} \left\{ \frac{1}{4} [\Phi_s, \bar{\Phi}_s]^2 \right\} \\ & + \frac{1}{2g_0^2} \sum_{\langle st \rangle \in L} \alpha_{\langle st \rangle} \text{Tr} \left\{ (U_{st} \Phi_t U_{st}^{-1} - \Phi_s) (U_{st} \bar{\Phi}_t U_{st}^{-1} - \bar{\Phi}_s) \right\} \\ & + \frac{1}{2g_0^2} \sum_{f \in F} \alpha_f \text{Tr} \left\{ Y_f (Y_f - i\beta_f \mu(U_f)) \right\}, \end{aligned} \quad (3.15)$$

$$\begin{aligned} S_f = & \frac{1}{2g_0^2} \sum_{s \in S} \alpha_s \text{Tr} \left\{ -\frac{1}{4} \eta_s [\Phi_s, \eta_s] \right\} \\ & + \frac{1}{2g_0^2} \sum_{\langle st \rangle \in L} \alpha_{\langle st \rangle} \text{Tr} \left\{ -i\lambda_{st} (U_{st} \eta_t U_{st}^{-1} - \eta_s) - \lambda_{st} \lambda_{st} (U_{st} \bar{\Phi}_t U_{st}^{-1} + \bar{\Phi}_s) \right\} \\ & + \frac{1}{2g_0^2} \sum_{f \in F} \alpha_f \left\{ -\chi_f [\Phi_f, \chi_f] + i\beta_f \chi_f (Q\mu(U_f)) \right\}. \end{aligned} \quad (3.16)$$

If we consider the torus discretization corresponding to the square lattice (3.3) and set  $\alpha_s = \alpha_{\langle st \rangle} = \alpha_f = \beta_f = 1$ , this action reproduces that of the lattice formulation of two-dimensional  $\mathcal{N} = (2, 2)$  supersymmetric Yang-Mills theory given in [23, 32].

We make a comment before closing this subsection. The construction of the discretized theory given above is based on abstract data (3.1) which includes such polygons that cannot be interpreted as a discretization of any Riemann surface<sup>6</sup>. Since our main purpose

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<sup>6</sup>The 3D cubic lattice is a typical example.

in this paper is to discretize the two-dimensional topological gauge field theory, we will implicitly restrict the polygons to discretized Riemann surfaces in the next section. However, it is worth noting that our construction is applicable to a wider class of discretized objects in principle.

### 3.2 Classical continuum limit

Let us next consider the tree-level continuum limit. To this end, we assume that the given decomposition is sufficiently fine to approximate a Riemann surface  $\Sigma_g$ . We first define the “lattice spacing” through the relation,

$$a^2 N_F = \int_{\Sigma_g} d^2x \sqrt{g(x)}, \quad (3.17)$$

where  $N_F$  is the number of faces. In other words,  $a^2$  is equal to the average area of the faces. The continuum limit is defined by the limit of  $a \rightarrow 0$  and  $N_F \rightarrow \infty$  while fixing (3.17). We also define the area of each face as

$$a^2 A_f = \int_{\sigma_f} d^2x \sqrt{g(x)}, \quad (3.18)$$

where the integration is taken over the region (simplex)  $\sigma_f$  corresponding to the face  $f$ . In particular, we see

$$a^2 \sum_{f \in F} A_f \rightarrow \int_{\Sigma_g} d^2x \sqrt{g(x)}, \quad (3.19)$$

in the continuum limit.

Since we assume that the given decomposition sufficiently well approximates the Riemann surface  $\Sigma_g$ , we can identify the index  $s$  of a site with a two-dimensional coordinate  $x_s$ . Then, corresponding to the link  $\langle st \rangle$ , we can define a covariant vector,

$$e_{st}^\mu \equiv \frac{1}{a} (x_t^\mu - x_s^\mu), \quad (3.20)$$

where  $x_s$  and  $x_t$  are the two-dimensional coordinates corresponding to the sites  $s$  and  $t$ , respectively. Here let  $L_f$  denote a set of links that construct the face  $f$ . From the definition of the continuum limit, it is natural to identify a face as a tangent space of the Riemann surface. Thus we assume that all the vectors  $e_{st}^\mu$  for  $\langle st \rangle \in L_f$  are in the same two-dimensional plane.

Here we should note that all the fields on a general lattice are defined as dimensionless quantities, thus we must supply appropriate powers of  $a$  in order to define the corresponding continuum fields. We must also require that the correspondence must be

consistent with the  $Q$ -transformation. From these requirements, it is natural to consider the following correspondence between the discrete and continuum fields:

$$\begin{aligned}\Phi_s &= a\Phi(x_s), \quad \bar{\Phi}_s = a\bar{\Phi}(x_s), \quad \eta_s = a^{\frac{3}{2}}\eta(x_s), \\ U_{st} &= e^{iae_{st}^\mu A_\mu(x_s + \frac{a}{2}e_{st}^\mu)}, \\ \lambda_{st} &= a^{\frac{3}{2}}e^{\frac{i}{2}ae_{st}^\mu A_\mu(x_s + \frac{a}{2}e_{st}^\mu)}e_{st}^\nu \lambda_\nu(x_s + \frac{a}{2}e_{st}^\mu)e^{-\frac{i}{2}ae_{st}^\mu A_\mu(x_s + \frac{a}{2}e_{st}^\mu)}, \\ Y_f &= a^2Y(x_f), \quad \chi_f = a^{\frac{3}{2}}\chi(x_f).\end{aligned}\tag{3.21}$$

Not only the fields but also the supercharge  $Q$  and the coupling constant  $g_0$  on the lattice are dimensionless as well. Therefore they must also be rescaled as

$$Q = a^{1/2}\hat{Q}, \quad \frac{1}{g_0^2} = \frac{1}{a^2g_{2d}^2}.\tag{3.22}$$

Let us now evaluate the action (3.8) in the continuum limit. Substituting (3.21) and (3.22) in the action (3.8), we obtain

$$S_S = \frac{\hat{Q}}{2g_{2d}^2} \sum_{f \in F} a^2 A_f \left( \sum_{s \in S_f} \frac{\alpha_s^f}{A_f} \text{Tr} \left( \frac{1}{4} \eta(x_s) [\Phi(x_s), \bar{\Phi}(x_s)] \right) \right),\tag{3.23}$$

$$S_L = \frac{\hat{Q}}{2g_{2d}^2} \sum_{f \in F} a^2 A_f \left( \sum_{\langle st \rangle \in L_f} \frac{\alpha_{\langle st \rangle}^f}{A_f} e_{st}^\mu e_{st}^\nu \text{Tr} \left\{ -i\lambda_\mu(x_s) \mathcal{D}_\nu \bar{\Phi}(x_s) + \mathcal{O}(a) \right\} \right),\tag{3.24}$$

$$S_F = \frac{\hat{Q}}{2g_{2d}^2} \sum_{f \in F} a^2 A_f \left( \frac{\alpha_f}{A_f} \text{Tr} \left\{ \chi(x_f) \left( Y(x_f) - i\beta_f A_f \frac{\epsilon^{\mu\nu}}{\sqrt{g(x_f)}} F_{\mu\nu} + \mathcal{O}(a) \right) \right\} \right),\tag{3.25}$$

where  $S_f$  is the set of sites that construct the face  $f$ ,  $F_s$  is the set of faces that meet at the site  $s$ ,  $\alpha_s^f$  and  $\alpha_{\langle st \rangle}^f$  are constants satisfying  $\alpha_s = \sum_{f \in F_s} \alpha_s^f$  and  $\alpha_{\langle st \rangle} = \sum_{f \in F_{\langle st \rangle}} \alpha_{\langle st \rangle}^f$ , respectively, and we have used

$$\mu(U_f) = ia^2 \frac{A_f}{\sqrt{g(x_f)}} \epsilon^{\mu\nu} F_{\mu\nu} + \mathcal{O}(a^3),\tag{3.26}$$

while evaluating (3.25) (see the appendix A). Here  $F_{\langle st \rangle}$  is the set of faces that share the link  $\langle st \rangle$ <sup>7</sup>. It is easy to see that the continuum limit of the site action (3.23) and the face action (3.25) becomes the corresponding part of the continuum action (2.9) by setting the parameters  $\alpha_s$ ,  $\alpha_f$  and  $\beta_f$  as

$$\alpha_s = \sum_{f \in F_s} \frac{A_f}{|S_f|}, \quad \alpha_f = A_f, \quad \beta_f = \frac{1}{A_f}.\tag{3.27}$$

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<sup>7</sup> If the link  $\langle st \rangle$  is a component of the boundary, if it exists, of the surface, only one face shares it. Otherwise two faces share it.

The link part (3.24) is slightly more complicated; in order to reproduce the continuum action,  $\alpha_{\langle st \rangle}$  must satisfy

$$\sum_{\langle st \rangle \in L_f} \alpha_{\langle st \rangle}^f e_{st}^\mu e_{st}^\nu = A_f g^{\mu\nu}(x_f). \quad (3.28)$$

It is easy to see that we can determine the value of  $\alpha_{\langle st \rangle}$  for any given Riemann surface by solving (3.28). In fact, when the face  $f$  consists of  $n$  links,  $l_i$  ( $i = 1, \dots, n$ ), the rank of the  $3 \times n$  matrix  $M_i^I \equiv e_{l_i}^\mu e_{l_i}^\nu$  ( $I = (\mu, \nu) = (1, 1), (2, 2), (1, 2)$ ) is three since we assume that all the vectors  $\vec{e}_{l_i}$  are in the same two-dimensional plane. In particular, if we consider triangulation,  $\alpha_{\langle st \rangle}^f$  are uniquely determined through the equation (3.28). Therefore we see that the classical continuum limit of the discretized theory (3.14) becomes two-dimensional topological field theory on the Riemann surface  $\Sigma_g$  by setting  $\alpha_s$ ,  $\alpha_{\langle st \rangle}$ ,  $\alpha_f$  and  $\beta_f$  as (3.27) and (3.28).

### 3.3 Radiative corrections

We next discuss possible radiative corrections that appear in taking the continuum limit. The discussion is completely parallel with that for Sugino's formulation given in [23–26]. From the power counting, we see that possible relevant or marginal operators that can appear radiatively are  $B_1(x)$  or  $B_1(x)B_2(x)$  with bosonic fields  $B_1(x)$  and  $B_2(x)$ . From the gauge symmetry and the  $\hat{Q}$ -symmetry, the only possible terms are  $\text{Tr } \Phi(x)$  and  $\text{Tr } \Phi(x)^2$  up to constant factors.

As announced, the situation differs depending on whether the scalar fields  $\Phi(x)$  and  $\bar{\Phi}(x)$  are complex conjugate with each other or not. When  $\Phi(x)$  and  $\bar{\Phi}(x)$  are complex conjugate with each other as in Sugino's formulation, both  $\text{Tr } \Phi(x)$  and  $\text{Tr } \Phi(x)^2$  are forbidden by the  $U(1)_R$  symmetry (2.10). Therefore, we do not need any fine-tuning in taking the continuum limit in this case. On the other hand, when  $\Phi(x)$  and  $\bar{\Phi}(x)$  are independent hermitian variables, there is no symmetry that forbids the appearance of these operators radiatively. Therefore we need to add counter-terms,

$$S_C = \begin{cases} \sum_{s \in S} \text{Tr } (c_1 \Phi_s^2 + c_2 \Phi_s) & \text{for } G = U(N), \\ \sum_{s \in S} \text{Tr } (c_1 \Phi_s^2) & \text{for } G = SU(N), \end{cases} \quad (3.29)$$

to the action and tune the parameters  $c_1$  ( $c_1$  and  $c_2$ ) for  $G = SU(N)$  ( $G = U(N)$ ) in taking the continuum limit<sup>8</sup>.

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<sup>8</sup>Because of the  $Q$ -symmetry, we see that the expectation values of some operators in  $Q$ -cohomology can be exactly evaluated even in the lattice theory [56]. In simulation, therefore, we will be able to use this exact result in tuning  $c_1$  and  $c_2$ .

## 4 Conclusion and discussion

In this paper, we have constructed a discrete formulation of the topologically twisted  $\mathcal{N} = (2, 2)$  supersymmetric Yang-Mills theory on an arbitrary two-dimensional lattice while preserving a supercharge. When the polygon decomposition (general lattice) is the discretization of the Riemann surface  $\Sigma_g$ , the continuum limit of this theory becomes the topologically twisted  $\mathcal{N} = (2, 2)$  supersymmetric Yang-Mills theory on  $\Sigma_g$ . If we consider the usual square lattice as an example of the decomposition, our model reproduces Sugino's lattice formulation of  $\mathcal{N} = (2, 2)$  supersymmetric Yang-Mills theory on the torus.

We have also shown that we can take the continuum limit without any fine-tuning if the theory possesses  $U(1)_R$  symmetry, i.e., we regard the two scalar fields in the vector multiplet as being complex conjugate with each other. On the other hand, if the scalar fields are independent Hermitian variables and the gauge group is  $SU(N)$  (or  $U(N)$ ), there is no  $U(1)_R$  symmetry in the model and we need one-parameter (or two-parameters) tuning in the continuum limit.

A natural question would arise as to whether there is a fermion doubler in this model or not. In order to answer this question, we have to examine if the kinetic terms of the fermions have no non-trivial zero, which depends on the structure of the discretization. However, we should recall that the origin of the fermion doubler is the periodicity in the momentum space, which is associated with the discrete translational invariance of lattice. Since a general lattice has less discrete translational symmetry than the usual square lattice, there is less chance for fermion doublers to appear. In addition, even if we consider the square lattice, it is shown that fermion doubler is absent [23]. Although it is still possible that fermion doublers appear by discretizing the Riemann surface by a highly symmetric tiling, we can conclude that there is no fermion doubler in most cases.

In the continuum theory, the so-called localization is used to examine the topological nature of the two-dimensional gauge theory [55]. Since our model preserves the scalar supersymmetry, which is the crucial symmetry in order that localization works, we can use the same technique in the discretized theory, which will be discussed separately in [56].

It will be straightforward to apply our method to the two-dimensional  $\mathcal{N} = (4, 4)$  and  $(8, 8)$  supersymmetric Yang-Mills theories or two-dimensional supersymmetric QCD. Furthermore our method is also applicable to the orbifold lattice theory [5–8]. The original orbifold lattice theory is based on the concept of deconstruction and is constructed by dividing a matrix theory (mother theory) by a discrete subgroup of the mother theory. The only background we can obtain in this way is the torus: it seems to be impossible that the standard orbifold projection constructs a theory on an arbitrary Riemann surface. On

the other hand, by using our method, we can construct the theory on the arbitrary lattice and we can embed the fields in sparse matrices. In this sense, our method can be regarded as a non-trivial extension of deconstruction, which will be connected with network theory. It might provide a novel way to examine the topological nature of gauge theory.

Including the fluctuation of polygons like Regge calculus [57] or dynamical triangulation [58] will be a fascinating next step. To this end, our set-up given in the section 3 would be insufficient to generate Riemann surface dynamically because it includes too-wide discretized objects. One plausible idea is to restrict the discretization to a simplicial complex. It will be interesting question to see if the diffeomorphism invariance is recovered in the continuum limit under such a restriction.

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## A Continuum limit of the plaquette variable

Let us consider a face  $(s_1, \dots, s_n)$  and the corresponding plaquette variable,

$$U_f = \prod_{i=1}^n U_{s_i s_{i+1}} \quad (s_{n+1} = s_1). \quad (\text{A.1})$$

We here assume that the vectors  $e_{s_k s_{k+1}}$  constructing this face span the same two-dimensional plane. Recalling that it is reasonable to think that the continuum gauge field is living at the middle point of the link:

$$U_{s_k s_{k+1}} = \exp \left\{ i a e_{s_k s_{k+1}}^\mu A_\mu \left( s_k + \frac{a}{2} e_{s_k s_{k+1}} \right) \right\}, \quad (\text{A.2})$$

and the argument of  $A_\mu$  is rewritten as

$$s_k + \frac{a}{2} e_{s_k s_{k+1}} = s_1 + \frac{a}{2} (e_{s_1 s_2} + e_{s_2 s_3} + \dots + e_{s_{k-1} s_k} - e_{s_{k+1} s_{k+2}} - \dots - e_{s_n s_1}), \quad (\text{A.3})$$

we can rewrite (A.2) as

$$U_{s_k s_{k+1}} = \exp \left\{ i a e_{st}^\mu A_\mu(s_1) + \frac{i}{2} a^2 e_{s_k s_{k+1}}^\mu \left( \sum_{l < k} e_{s_l s_{l+1}}^\nu - \sum_{l > k} e_{s_l s_{l+1}}^\nu \right) \partial_\nu A_\mu(s_1) + \mathcal{O}(a^3) \right\}. \quad (\text{A.4})$$

Substituting (A.4) to (A.1) and using the Campbell-Baker-Hausdorff formula,

$$e^{M_1} e^{M_2} \dots e^{M_n} = e^{\sum_{i=1}^n M_i + \frac{1}{2} \sum_{i < j} [M_i, M_j] + \dots}, \quad (\text{A.5})$$

we see

$$U_f = \exp \left\{ \frac{i}{2} a^2 C_f^{\mu\nu} F_{\mu\nu}(s_1) + \mathcal{O}(a^3) \right\}, \quad (\text{A.6})$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \quad (\text{A.7})$$

and

$$C_f^{\mu\nu} = \frac{1}{2} \sum_{k=1}^n e_{s_k s_{k+1}}^\mu \left( - \sum_{l < k} e_{s_l s_{l+1}}^\nu + \sum_{l > k} e_{s_l s_{l+1}}^\nu \right). \quad (\text{A.8})$$

In order to see the geometrical meaning of  $C_f^{\mu\nu}$ , it is convenient to rewrite it as

$$C_f^{\mu\nu} = \frac{1}{2} \sum_{i=3}^n \left( e_{s_i s_{i-1}}^\mu e_{s_i s_1}^\nu - e_{s_i s_{i-1}}^\nu e_{s_i s_1}^\mu \right), \quad (\text{A.9})$$

where  $e_{s_i s_1} \equiv -e_{s_i s_{i+1}} - e_{s_{i+1} s_{i+2}} - \dots - e_{s_n s_1}$ . Since  $\frac{1}{2}(e_{s_i s_{i-1}}^1 e_{s_i s_1}^2 - e_{s_i s_{i-1}}^2 e_{s_i s_1}^1)$  is the area of the triangle with the vertices  $s_1, s_{i-1}, s_i$ , we see

$$C_f^{\mu\nu} = \frac{A_f}{\sqrt{g(x_f)}} \epsilon^{\mu\nu}, \quad (\text{A.10})$$

which is proportional to a unit area of the polygon made up of  $e_{s_i s_{i+1}}$ 's.

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